Department of Mathematics & Philosophy of Engineering

Faculty of Engineering Technology

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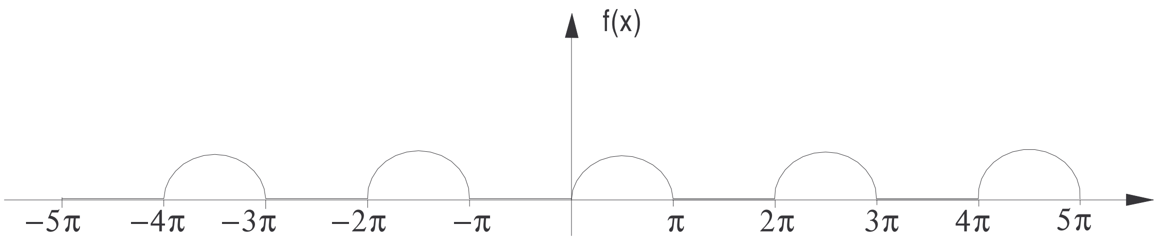
Course: MPZ 3132-Engineering Mathematics IB **Model Answer No.01 Academic Year 2011/2012**

1. 1.1 Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The Fourier series or Fourier expansion corresponding to $f(x)$ is defined to be $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{L} + b_n Sin\frac{n\pi x}{L}\right)$.

1.1.1



1.1.2

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}\sin x dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi} SinxCosnxdx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[Sin(n+1)x - Sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[Sin(n+1)x - Sin(n-1)x\right]dx = \frac{1}{2\pi}\left[\frac{-Cos(n+1)x}{n+1} + \frac{Cos(n-1)x}{n-1}\right]_0^{\pi} \quad n \neq 1$$

$$=\frac{1}{2\pi}\left[\frac{-(1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{1}{2\pi}\left[\frac{-(-1)^{n-1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{-2}{n^2-1}\right]=\frac{1}{2\pi}\frac{2}{n^2-1}\left(-1+(-1)^{n-1}\right)$$

$$a_1=\frac{1}{\pi}\int_0^\pi Sinx\ Sinx\ dx=\frac{1}{\pi}\int_0^\pi\frac{(1-\cos 2x)}{2}dx$$

$$=\frac{1}{2\pi}\left[x-\frac{Sin2x}{2}\right]_0^\pi=\frac{1}{2\pi}\pi=\frac{1}{2}$$

$$a_1=\frac{1}{\pi}\int_0^\pi Sinx\ Cosx\ dx=\frac{1}{2\pi}\int_0^\pi Sin2xdx=\frac{1}{2\pi}\left[Cos2x\right]_0^\pi=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^\pi f(x)Sinnxdx=\frac{1}{\pi}\int_0^\pi SinxSinnxdx=\frac{1}{2\pi}\int_0^\pi 2SinnxSinxdx$$

$$b_n=\frac{1}{2\pi}\left[\int_0^\pi Cos(n-1)x-Cos(n+1)x\right]dx=\frac{1}{2\pi}\left[\frac{Sin(n-1)x}{n-1}-\frac{Sin(n+1)x}{n+1}\right]_0^\pi=0$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^\infty(a_nCosnx+b_nSinnx)=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=2}^\infty\left(\frac{-1+(-1)^{n-1}}{n^2-1}\right)Cosnx+\frac{1}{2}Sinx$$

If n is even $n=2r\ \ r\in\mathbb{Z}^+$

If n is odd $n=2r+1\ \ r\in\mathbb{Z}^+$

$$a_{2r+1}=\frac{-1-1}{(2r+1)^2-1}=0\ r\in\mathbb{Z}^+$$

$$a_{2r}=\frac{-1-1}{4r^2-1}=\frac{-2}{4r^2-1}\ \ r\in\mathbb{Z}^+$$

$$f(x)=\frac{1}{\pi}+\frac{1}{2}Sinx-\frac{2}{\pi}\sum_{r=1}^\infty\frac{Cos2rx}{4r^2-1}$$

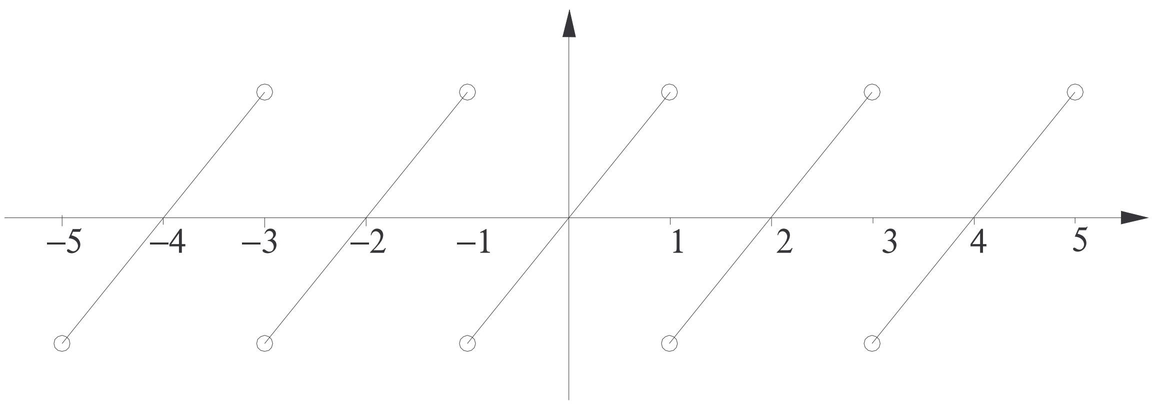
1.1.3

$$f(\pi/2)=\frac{1}{\pi}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{(-2)}{4r^2-1}Cos\pi r-\frac{1}{2}$$
$$1=\frac{1}{\pi}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{(-2)(-1)^r}{4r^2-1}+\frac{1}{2}$$
$$\frac{1}{2}-\frac{1}{\pi}=-2\sum_{r=1}^{\infty}\frac{(-1)^r}{4r^2-1}$$
$$\frac{\pi-2}{4}=\sum_{r=1}^{\infty}\frac{(-1)^r}{4r^2-1}$$

1.2 $g(x)=x,\ x\in(-1,1)$

$g(x)=g(x+2h)\ h\in\mathbb{Z}$

1.2.1



$g(-x)=(-x)=-g(x)\quad \therefore g$ is a odd function.

$\therefore \mathrm{a}_n=0\ \ \forall n\in\mathbb{Z}_0^+$

$$b_n=\frac{1}{1}\int_{-1}^{1}f(x)Sin\frac{n\pi x}{1}dx=2\int_{0}^{1}xSinn\pi xdx=2\left[-\frac{1}{n\pi}xConn\pi x+\frac{1}{n^2\pi^2}Sinn\pi x\right]_0^1$$
$$b_n=2\left[-\frac{1}{n\pi}(1)(-1)^n\right]=\frac{2(-1)^{n+1}}{n\pi}=\sum_{r=1}^{\infty}2\frac{(-1)^{r+1}}{r\pi}Sinr\pi x$$

Substituting values for the defined Fourier series

$$g(x)=\sum_{r=1}^{\infty}2\frac{(-1)^{r+1}}{r\pi}Sinr\pi x$$

1.2.3

` Integrating both sides

$$\frac{x^2}{2}+C=\sum_{r=1}^{\infty}2\frac{(-1)^{r+1}}{r+1}(\frac{-Cosr\pi x}{r\pi})$$

When $x=0$

$$C=\sum_{r=1}^{\infty}2\frac{(-1)^{r+1}}{r^2\pi^2}=\frac{1}{\pi^2}\sum_{r=1}^{\infty}2\frac{(-1)^r}{r^2}$$

$$\frac{x^2}{2}=\sum_{r=1}^{\infty}2\frac{(-1)^r}{r^2\pi^2}Cosr\pi x-\frac{1}{\pi^2}\sum_{r=1}^{\infty}2\frac{(-1)^r}{r}$$

$$x^2=\frac{4}{\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^r}{r^2}\left(Cosr\pi x-1\right)$$

2.

Suppose that

(i) $f(x)$ is defined and single-value except possible at a finite number of points in $(-L,L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L,L)$

2.1

$$f(x)=x(x+1) \quad x\in(-\pi,\pi)$$

$$f(x)=f(x+2h\pi)$$

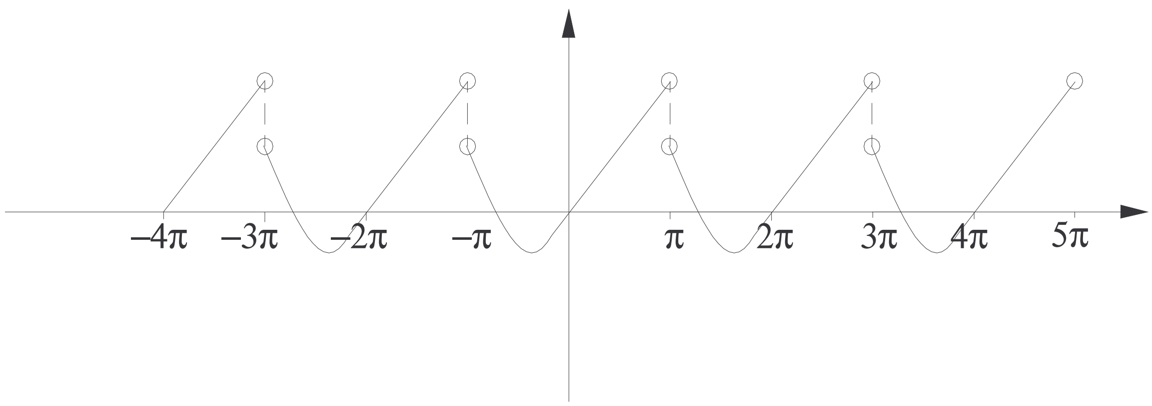
$$f(0)=0,\ f(-1)=0$$

$$f(x)=(x+\frac{1}{2})^2-\frac{1}{4}$$

$$\min f(x)=-\frac{1}{4} \quad \text{and } x=-\frac{1}{2}x$$

$$x\xrightarrow{\lim}\pi^- f(x)=\pi^2+\pi,\ x\xrightarrow{\lim}\pi^+ f(-x)=\pi$$

2.1.1



2.1.2.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)dx = \frac{1}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)Cosnxdx = \frac{2}{\pi}\int_{0}^{\pi}x^2 Cosnxdx \quad \left(xcoxnx \text{ is odd and } x^2\cos nx \text{ is even}\right)$$

$$a_n = \frac{2}{\pi}\left[\frac{1}{n}x^2 Sinnx + \frac{1}{n^2}2xCoxnx - \frac{1}{n^3}2Sinnx\right]_0^{\pi} = \frac{2}{\pi}2\pi\frac{(-1)^n}{n^2} = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)Sinnxdx = \frac{2}{\pi}\int_{0}^{\pi}xSinnxdx$$

$$\left(\therefore\ x^2 Sinx \text{ is odd } \ \& \ x\sin nx \text{ is even}\right)$$

$$b_n = \frac{2}{\pi}\left[-\frac{1}{n}xCosnx + \frac{1}{n^2}Sinnx\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{1}{n}\pi(-1)^n\right] = -\frac{2}{n}(-1)^n$$

$$x(x+1) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}Consnx - \frac{2}{n}(-1)^n Sinnx$$

$$x(x+1) = \frac{\pi^2}{3} + \sum_{r=1}^{\infty}2(-1)^r\left(\frac{4}{r^2}Cosrx - \frac{1}{r}Sinrx\right)$$

2.1.3

$$x \xrightarrow{\lim} \pi^+ f(x) = \pi^2 - \pi, \quad x \xrightarrow{\lim} \pi^- f(x) = \pi^2 + \pi$$

$$\frac{1}{2}\left[x \xrightarrow{\lim} \pi^+ f(x) + x \xrightarrow{\lim} \pi^- f(x) \right] = \frac{\pi^2}{3} + \sum_{r=1}^{\infty} 2(-1)^r \frac{2}{r^2}(-1)^r$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$\frac{\pi^2}{6} = \sum_{r=1}^{\infty} \frac{1}{r^2}$$

2.2.1

$$g(x) = \begin{cases} \frac{1}{2} - x & -1/2 < x < 0 \\ \frac{1}{2} + x & 0 < x < 1/2 \end{cases}$$

When $-1/2 < x < 0 \quad g(x) = \frac{1}{2} - x$

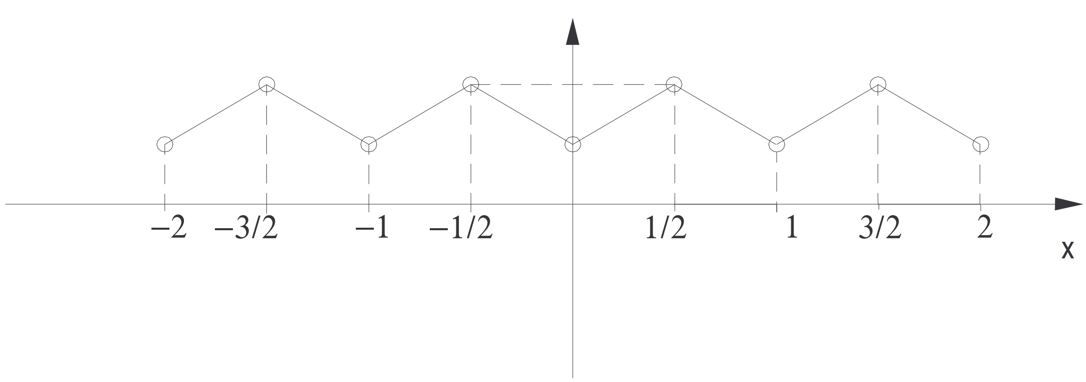
$0 < x < 1/2 \quad g(-x) = \frac{1}{2} + (-x) = \frac{1}{2} - x = g(x)$

When $0 < x < 1/2 \quad g(x) = \frac{1}{2} + x$

$-1/2 < -x < 0 \quad g(-x) = \frac{1}{2} - (-x) = \frac{1}{2} + x = g(x)$

$g(x) = g(-x) \quad \therefore g$ is an even function.

2.2.2



$$a_0 = \frac{2}{(1/2)}\int_0^1 (\frac{1}{2}+x)dx = 4\left[\frac{1}{2}x+\frac{x^2}{2}\right]_0^{1/2} = 4\left[\frac{1}{4}+\frac{1}{8}\right] = 1+\frac{1}{2} = \frac{3}{2}$$

$$a_n = \frac{2}{(1/2)}\int_0^{1/2} (\frac{1}{2}+x)Cos\left(\frac{n\pi x}{1/2}\right)dx = 4\int_0^{1/2} (\frac{1}{2}+x)Cosn\pi xdx$$

$$a_n = 4\left[\frac{1}{9n\pi}(1/2+x)Sin2n\pi x+\frac{1}{4n^2\pi^2}Cos2n\pi x\right]_0^{1/2}$$

$$a_n = \frac{4}{4n^2\pi^2}\left[Cosn\pi-1\right] = \frac{4}{n^2\pi^2}\left((-1)^n-1\right)$$

Since g is even $b_n = 0$.

$$\therefore g(x) = \frac{3}{4}+\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left((-1)^n-1\right)Cos2n\pi x$$

n is even $n = 2r \quad r \in \mathbb{Z}^+$

$a_{2r} = 0$

n is odd $n = 2r-1 \;\; r \in \mathbb{Z}^+$

$$g(x) = \frac{3}{4}+\frac{1}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}(-1-1)Cos(2r-1)\pi x$$

$$g(x) = \frac{3}{4}-\frac{2}{\pi^2}\sum_{r=1}^{\infty}\frac{Cos2(2r-1)\pi x}{(2r-1)^2}$$

$$g(0) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{3}{4}-\frac{2}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}$$

$$-\frac{1}{4} = \frac{-2}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}$$

$$\therefore \frac{\pi}{8} = \sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}$$

3.

3.1

3.1.1 $f_1(x) = \begin{cases} \pi - x & x \in (0,\pi) \\ -x-\pi & x \in (-\pi,0) \end{cases}$

And $f_1(x) = f_1(x+2k\pi) \quad k \in \mathbb{Z}$

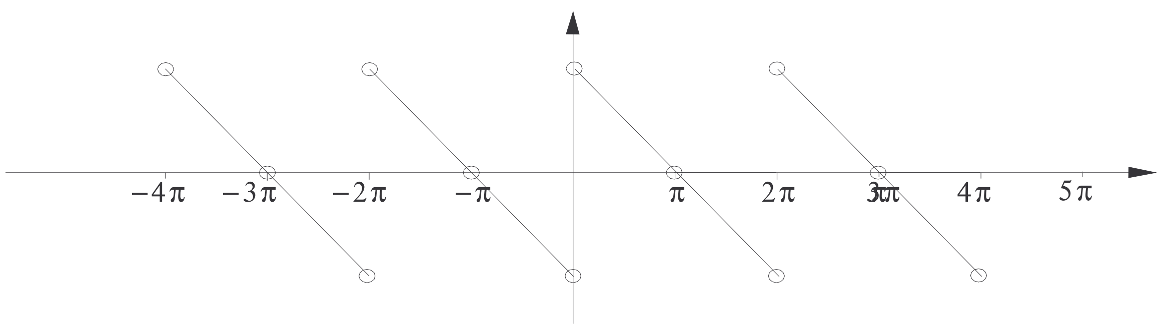
3.1.2 $f_2(x) = \begin{cases} \pi - x & x \in (0,\pi) \\ x+\pi & x \in (-\pi,0) \end{cases}$

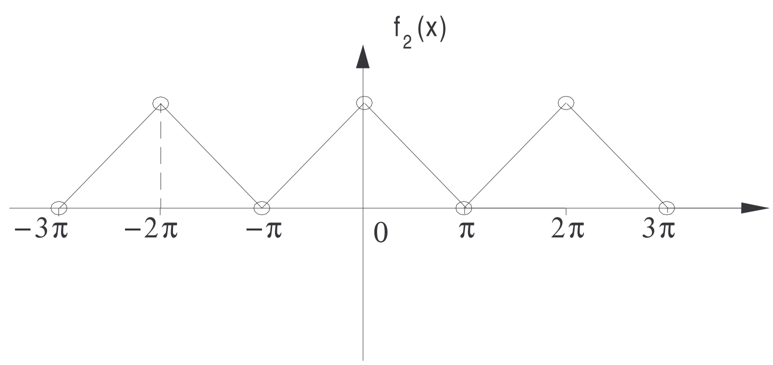
And $f_1(x) = f_1(x+2k\pi) \quad k \in \mathbb{Z}$

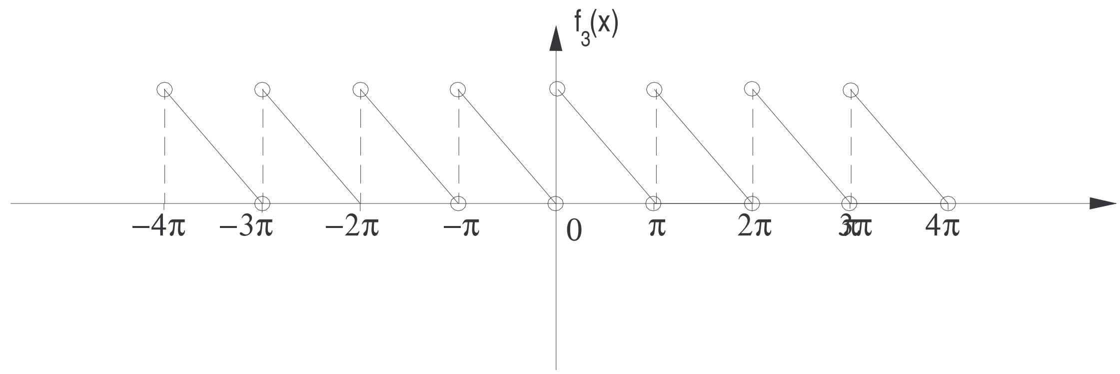
3.1.3 $f_3(x) = \pi - x \quad x \in (0,\pi)$

And $f_3(x) = f_3(x+k\pi) \quad k \in \mathbb{Z}$

3.2







3.3

3.3.1 $f_1(x)=\begin{cases}\pi-x & x\in(0,\pi)\\ -\pi-x & x\in(-\pi,0)\end{cases}$

This function is odd. $\therefore\ a_0=0,\ \mathrm{a}_n=0$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)Sinnxdx=\frac{2}{\pi}\int_0^{\pi}(\pi-x)Sinnxdx$$

$$=-\frac{2}{\pi}\left[-\frac{(\pi-x)}{n}Cosnx+\frac{(-1)}{n^2}Sinnx\right]_0^{\pi}=-\frac{2}{\pi}\left[\frac{-\pi}{n}\right]=\frac{2}{n}$$

$$\therefore \pi-x=\sum_{r=1}^{\infty}\frac{2Sinrx}{r}=2\sum_{r=1}^{\infty}\frac{Sinrx}{r}$$

3.3.2

$$f_2(x)=\begin{cases}\pi-x & \mathrm{x}\in(0,\pi)\\ x+\pi & x\in(-\pi,0)\end{cases}$$

Since $f_2(x)$ is even $b_n=0,\ a_0=\frac{2}{\pi}\int_0^{\pi}(\pi-x)dx=\frac{2}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\left[\pi^2-\frac{\pi^2}{2}\right]=\pi$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)Cosnxdx=\frac{2}{\pi}\left[\frac{1}{n}(\pi-x)Sinax-\frac{1}{n^2}Cosnx\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{1}{n^2}\left((-1)^n-1\right)\right]$$

n is odd $n=2r-1\quad r=1,2,...$

$$a_{2r-1}=\frac{2}{\pi}\left[\frac{1}{(2r-1)^2}(-1-1)\right]=\frac{-4}{\pi(2r-1)^2}$$

n is even

$$a_{2r} = 0 \quad r = 1, 2, \ldots.$$

$$\pi - x = \frac{\pi}{2} + \sum_{r=1}^{\infty} \frac{-4}{\pi(2r-1)^2} Cos(2r-1)x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{Cos(2r-1)x}{(2r-1)^2}$$

3.3.3

$$f_3(x) = \pi - x \quad x \in (0, \pi)$$
$$f_3(x) = f(x + k\pi x)$$

$$a_0 = \frac{1}{\pi/2} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \pi$$

$$a_n = \frac{1}{\pi/2} \int_0^{\pi} (\pi - x) Cos 2\pi x dx = \frac{2}{\pi} \left[\frac{(\pi - x)}{2n} Sin 2nx + \frac{(-1)}{4n^2} Cos 2nx \right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi/2} \int_0^{\pi} (\pi - x) Sin 2nx dx = \frac{2}{\pi} \left[-\frac{(\pi - x)}{2n} Cos 2nx + \frac{1}{4n^2} (-1) \sin 2nx \right]_0^{\pi} = \frac{2}{\pi} \left(\frac{\pi}{2n} \right) = \frac{1}{n}$$

$$\therefore \pi - x = \frac{\pi}{2} + \sum_{r=1}^{\infty} \frac{Sin 2rx}{r}$$

3.3.3

$$\int_0^{2c} (f(x))^2 dx = c \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$
$$\int_0^{c} (f(x))^2 dx = \frac{c}{2} \left[\frac{a_0^2}{2} + \sum_{r=1}^{\infty} a_n^2 \right] = \frac{c}{2} \sum_{r=1}^{\infty} b_n^2$$
$$\int_0^{\pi} (\pi - x)^2 dx = \frac{\pi}{2} \left[\frac{\pi^2}{4} + \frac{16}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^4} \right]$$
$$\left[\frac{-1(\pi - x)^3}{3} \right]_0^{\pi} = \frac{\pi}{2} \left[\frac{\pi^2}{4} + \frac{16}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^4} \right]$$

$$\frac{\pi^3}{3}=\frac{\pi}{2}\left(\frac{\pi^2}{4}+\frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}\right)$$

$$\frac{\pi^2}{3}-\frac{\pi^2}{4}=\frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$$

$$\frac{\pi^4}{96}=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$$

$$\int_0^{\pi}(\pi-x)^2dx=\frac{\pi}{2}\sum_{r=1}^{\infty}\frac{1}{r^2}+\frac{\pi^2}{4}$$

$$\frac{\pi^2}{3}=\frac{\pi}{2}\left(\sum_{r=1}^{\infty}\frac{1}{r^2}+\frac{\pi^2}{4}\right)$$

$$\frac{\pi^2}{3}-\frac{\pi^2}{4}=\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^2}$$

$$\frac{\pi^2}{12}=\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^2}$$

$$\frac{\pi^2}{6}=\sum_{r=1}^{\infty}\frac{1}{r^2}$$

4. Let the function f be n times differentiable at a. Then we define nth Taylor polynomial for f about $x=a$ to be

$$f_x(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+...+\frac{f^n(x)}{n!}(x-a)^n$$

4.1

4.1.1

$$\frac{dy}{dx}=\frac{(2x-2Cos\theta)}{(1-2xCos\theta+x^2)}$$

$$(1-2xCos\theta+x^2)\frac{dy}{dx}=(2x-2Cos\theta)$$

$$(1-2xCos\theta+x^2)\frac{d^2y}{dx^2}+(2x-2Cos\theta)\frac{dy}{dx}=2$$

$$(1-2xCos\theta+x^2)\frac{d^3y}{dx^3}+2(2x-2Cos\theta)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$
$$(1-2xCos\theta+x^2)\frac{d^4y}{dx^4}+3(2x-2Cos\theta)\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}=0$$
$$(1-2xCos\theta+x^2)\frac{d^5y}{dx^5}+4(2x-2Cos\theta)\frac{d^4y}{dx^4}+12\frac{d^3y}{dx^3}=0$$

4.1.2

$$f(0)=0,\ \ f^{'}(0)=-2Cos\theta,$$
$$f^2(0)=-(-2Cos\theta)(-2Cos\theta)+2=-2(2Cos^2\theta-1)=-2Cos2\theta$$

$$f^3(0)=-2(-2Cos\theta)(-2Cos2\theta)-2(-2Cos\theta)=-4(2Cos2\theta Cos\theta-Cos\theta)=-4(Cos\theta)$$

$$f^4(0)=-3(-2Cos\theta)(-4Cos3\theta)-6(-2Cos2\theta)=12(2Cos3\theta Cos\theta-Cos2\theta)=-12Cos4\theta$$

$$f^5(0)=-4(-2Cos\theta)(-12Cos4\theta)-12(-4Cos3\theta)=-48\left[2Cos4\theta Cos\theta-Cos3\theta\right]=48Cos5\theta$$

$$f(x)\approx f^4(0)+\frac{f^1 0)}{1!}x+\frac{f^2(0)}{2!}x^2+\frac{f^3(0)}{3!}x^3+\frac{f^4(0)}{4!}x^4+\frac{f^5(0)}{5!}x^5$$
$$=-2Cos2\theta x-2\frac{Cos2\theta}{2!}x^2-4\frac{Cos3\theta}{3!}x^3-12\frac{Cos4\theta}{4!}x^4-48\frac{Cos5\theta}{5!}x^5$$
$$=-2\left(Cos\theta x+\frac{Cos2\theta}{2}x^2+\frac{Cos3\theta}{3}x^3+\frac{Cos4\theta}{4}x^4+\frac{Cos5\theta}{5}x^5\right)$$
$$=-2\sum_{r=1}^{5}\frac{Cosr\theta}{r}x^r$$

4.2

$$f(x)=\ln(Cosx)$$
$$\frac{df(x)}{dx}=\frac{1}{Cosx}(-Sinx)=-\tan x$$
$$\frac{d^2f(x)}{dx^2}=-Sec^2x=-1-\tan^2 x=-1-\left(\frac{df(x)}{dx}\right)^2$$
$$\frac{d^3f(x)}{dx^3}=-2\frac{df(x)}{dx}\frac{d^2f(x)}{dx^2}$$
$$\frac{d^3f(x)}{dx^3}\neq 2\frac{df(x)}{dx}\frac{d^2f(x)}{dx^2}=0$$

4.2.1

$$\frac{d^4 f(x)}{dx^4}+2\frac{df(x)}{dx}\frac{d^3 f(x)}{dx^3}+2\left(\frac{d^2 f(x)}{dx^2}\right)^2=0$$

$$f(0)=0,\quad \frac{df(x)}{dx}=0,\quad \frac{df^2(x)}{dx^2}=-1,\quad \frac{df^3(x)}{dx^3}=0,\quad \frac{df^4(x)}{dx^4}=-2$$

The Taylor polynomial of order 4 is

$$= f(0)+\frac{1}{1}xf^{'}(0)+\frac{x}{2!}f^2(0)+\frac{x^3}{3!}f^3(0)+\frac{x^4}{4!}f^4(0)=-\frac{x^2}{2}-\frac{1}{4!}2x^4=-\frac{1}{2}x^2-\frac{1}{12}x^4$$

4.1.1

$$\ln(Cosx)\approx-\frac{x^2}{2}-\frac{1}{12}x^4$$
$$\ln(Cos\pi/4)\approx\frac{\pi^2}{16.2}-\frac{1}{12}\frac{\pi^4}{196}$$
$$-\ln\sqrt{2}=-\frac{\pi^2}{2.16}-\frac{1}{12}\frac{\pi^4}{196}$$
$$\frac{1}{2}\ln 2=\frac{\pi^2}{2.16}+\frac{1}{12}\frac{\pi^4}{196}$$
$$\ln 2=\frac{\pi^2}{2}\left(1+\frac{\pi^2}{96}\right)$$

5. The Taylor series of function $f(x)$ is given about $x=a$ defined as $f(x)=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$.

5.1

$$f(x)=Cosx\qquad f^{'}(x)=-Sinx$$
$$f^2(x)=-Cosx\quad f^3(x)=Sinx$$
$$f^4(x)=Cosx\qquad f^5(x)=-Sinx$$
$$f^6(x)=-Cosx\quad f^7(x)=Sinx$$

$$f^{2n}(x)=(-1)^n Cosx\quad f^{2n+1}(x)=(-1)^n Sinx\quad n\in\mathbb{Z}_0^+$$

$$f^{2n}(0)=(-1)^n\quad \&\quad f^{2n+1}(0)=0$$
$$\therefore\; Cosx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$f(x) = Sinx \qquad f^{'}(x) = Cosx$$
$$f^{2}(x) = -Sinx \quad f^{3}(x) = -Cosx$$
$$f^{4}(x) = Sinx \qquad f^{5}(x) = Cosx$$
$$f^{6}(x) = -Sinx \quad f^{7}(x) = -Cosx$$

$$f^{2n}(x) = (-1)^n Sinx \quad f^{2n+1}(x) = (-1)^n Cosx \quad n \in \mathbb{Z}_0^+$$

$$f^{2n}(0) = 0 \ \ \& \ \ f^{2n+1}(0)=(-1)^n$$
$$\therefore \ Sinx = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$$

5.1.1

$$Sin^2 x = \frac{1-Cosx}{2}$$
$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^n}{2n!} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{2n!}$$
$$Cos^2 x = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^n}{(2n+1)!}$$

5.1.2

$$Sin^2 x = \sum_{r=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\int_0^x Sinx^2 dx = \sum_{r=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+1)!}$$
$$Sinx^2 = \sum_{r=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$
$$Sinx = \sum_{r=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$
$$\int_0^x Sinx^2 dx = \left[\sum_{r=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} \right]_0^x$$

5.2

5.2.1

$$f(x)=\frac{1}{x}=x^{-1} \quad f^{1}(x)=(-1)x^{-2} \quad f^{2}(x)=(-1)(-2)x^{-3}$$

$$f^{3}(x)=(-1)(-2)(-3)x^{-4}$$

$$f^{n}(x)=(-1)(-2)(-3)\ldots(-n)x^{-n} \quad \forall n\in \mathbb{Z}_0^+$$

$$f^{n}(x)=\frac{n!(-1)^n}{3^{n+1}}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(3)(x-3)^n}{n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n n!(x-3)^n}{3^{n+1}n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n (x-3)^n}{3^{n+1}}$$

5.2.2

The above series is a geometric series with the common ratio $\frac{3-x}{3}$ and the first term $\frac{1}{3}$. The series converges if and only if $\left|\frac{3-x}{3}\right|<1 \quad 0<x<3$. Therefore, the convergence is $0<x<3$. The series converges to $\frac{1}{3}\frac{1}{1-\frac{(3-x)}{3}}=\frac{1}{x}$.